Negative Probabilities in Financial Modeling

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Abstract

We first define and derive general properties of negative probabilities. We then show how negative probabilities can be applied to modeling financial options such as Caps, Floors and Swapoptions. In trading practice, these options are typically valued in a Black-Scholes-Merton framework assuming a log-normal distribution for the underlying interest rate. However, in some cases, such as the 2008/2009 financial crisis, interest rates can get negative. Then the log-normal distribution is inapplicable. We show how negative probabilities can be applied to value interest rate options in a log-normal framework implying a positive probability for negative interest rates. A model in VBA, which prices Caps, Floors and Swapoptions with negative probabilities, is available upon request.

Key words: Negative probabilities, negative interest rates, Caps, Floors, Swaptions

JEL Classification: C10

1. Introduction

The classical probability theory is applied in most sciences and in many of the humanities. In particular, it is successfully used in physics and finance. However, physicists found that they need a more general approach than the classical probability theory. The first was Eugene Wigner (1932), who introduced a function, which looked like a conventional probability distribution and has later been better known as the Wigner quasi-probability distribution because in contrast to conventional probability distributions, it took negative values, which could not be eliminated or made nonnegative. The importance of Wigner's discovery for foundational problems was not recognized until much later. Another outstanding physicist, Dirac (1942) not only supported Wigner's approach but also introduced the physical concept of negative energy. He wrote:

"*Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative of money*."

After this, negative probabilities little by little have become popular although questionable techniques in physics. Bartlett (1945) worked out the mathematical and logical consistency of negative probabilities. However, he did not establish rigorous foundation for negative probability utilization. Khrennikov (2009) provides the first mathematical theory of negative probabilities in his textbook. However, he is doing this not in the conventional setting of real numbers but in the framework of *p*-adic analysis.

Negative probabilities are also used in mathematical finance. The concept of risk-neutral or pseudo probabilities is a popular concept and has been numerously applied, for example, in credit modeling by Jarrow and Turnbull (1995), and Duffie and Singleton (1999). Haug (2007) extends the risk-neutral framework to allow negative probabilities and shows how negative probabilities can help add flexibility to financial modeling.

The remaining paper is organized as follows. In section 2, we resolve the mathematical issue of the negative probability problem. We build a mathematical theory of extended probability as a probability function, which is defined for real numbers and can take both positive and negative values. Thus, extended probabilities include negative probabilities. Different properties of extended probabilities are found. In section 3, we give examples of negative nominal interest rates in financial practice and show problems of current financial modeling of negative interest rates. In Section 4, we build mathematical models of interest rate options as Caps and Floors and Swapoptions, integrating extended probabilities into the pricing model to allow for negative interest rates. Conclusions are given in Section 5. A follow up paper will specify >1 probabilities and apply them to financial options.

2. Mathematical theory of extended probability

Extended probabilities generalize the standard definition of a probability function. At first, we define extended probabilities in an axiomatic way and then develop application of extended probabilities to finance.

To define extended probability, EP, we need some concepts and constructions, which are described below. Some of them are well-known, such as, for example, set algebra, while others, such as, for example, random antievents, are new.

We remind that if X is a set, then $|X|$ is the number of elements in (cardinality of) X (Kuratowski and Mostowski, 1967). If $A \subseteq X$, then the complement of A in X is defined as $C_X A = X \setminus A$.

A system **B** of sets is called a *set ring* (Kolmogorov and Fomin, 1989) if it satisfies conditions (R1) and (R2):

(R1) $A, B \in \mathbf{B}$ implies $A \cap B \in \mathbf{B}$.

(R2) $A, B \in \mathbf{B}$ implies $A \Delta B \in \mathbf{B}$ where $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

For any set ring **B**, we have $\varnothing \in \mathbf{B}$ and $A, B \in \mathbf{B}$ implies $A \cup B, A \setminus B \in \mathbf{B}$.

Indeed, if $A \in \mathbf{B}$, then by R1, $A \setminus A = \varnothing \in \mathbf{B}$. If $A, B \in \mathbf{B}$, then $A \setminus B = ((A \setminus B) \cup (B \setminus A)) \cap A \in \mathbf{B}$. If $A, B \in \mathbf{B}$ and $A \cap B = \varnothing$, then $A \Delta B = A \cup B \in \mathbf{B}$. It implies that $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B) \in \mathbf{B}$. Thus, a system **B** of sets is a set ring if and only if it is closed with respect union, intersection and set difference.

Example 2.1. The set **CI** of all closed intervals $[a, b]$ in the real line $\boldsymbol{R}$ is a set ring.

Example 2.2. The set **OI** of all open intervals (a, b) in the real line $\boldsymbol{R}$ is a set ring.

A set ring **B** with a unit element, i.e., an element E from **B** such that for any A from **B** , we have $A \cap E = A$, is called a *set algebra* (Kolmogorov and Fomin, 1989).

Example 2.3. The set **BCI** of all closed subintervals of the interval $[a, b]$ is a set algebra.

Example 2.4. The set **BOI** of all open subintervals of the interval $[a, b]$ is a set algebra.

A set algebra **B** closed with respect to complement is called a *set field*.

Let us consider a set Ω, which consists of two irreducible parts (subsets) Ω^+ and Ω^-, i.e., neither of these parts is equal to its proper subset, a set **F** of subsets of Ω, and a function P from **F** to the set $\boldsymbol{R}$ of real numbers.

Elements from **F**, i.e., subsets of Ω that belong to **F**, are called *random events*.

Elements from $\mathbf{F}^+ = \{X \in \mathbf{F};\ X \subseteq \Omega^+\}$ are called *positive random events*.

Elements from Ω^+ that belong to $\mathbf{F}^+$ are called *elementary positive random events* or simply, *elementary positive random events*.

If $w \in \Omega^+$, then $-w$ is called the *antievent* of w.

Elements from Ω^- that belong to $\mathbf{F}^-$ are called *elementary negative random events* or *elementary random antievents*.

For any set $X \subseteq \Omega^+$, we define

$$X^+ = X \cap \Omega^+,$$

$$X^- = X \cap \Omega^-,$$

$$-X = \{-w;\ w \in X\}$$

and

$$\mathbf{F}^- = \{-A;\ A \in \mathbf{F}^+\}$$

If $A \in \mathbf{F}^+$, then $-A$ is called the *antievent* of A.

Elements from $\mathbf{F}^-$ are called *negative random events* or *random antievents*.

Definition 1. The function P from **F** to the set $\boldsymbol{R}$ of real numbers is called a *probability function*, if it satisfies the following axioms:

EP 1 (*Order structure*). There is a graded involution $\alpha: \Omega \to \Omega$, i.e., a mapping such that α^2 is an identity mapping on Ω with the following properties: $\alpha(w) = -w$ for any element w from Ω, $\alpha(\Omega^+) \supseteq \Omega^-$, and if $w \in \Omega^+$, then $\alpha(w) \notin \Omega^+$.

EP 2 (*Algebraic structure*). $\mathbf{F}^+ \equiv \{X \in \mathbf{F};\ X \subseteq \Omega^+\}$ is a set algebra that has Ω^+ as a member.

EP 3 (*Normalization*). $P(\Omega^+) = 1$.

EP 4 (*Composition*) $\mathbf{F} \equiv \{X;\ X^+ \subseteq \mathbf{F}^+\ \&\ X^- \subseteq \mathbf{F}^-\ \&\ X^+ \cap -X^- \equiv \varnothing\ \&\ X^- \cap -X^+ \equiv \varnothing\}$.

EP 5 (*Finite additivity*)

$$P(A \cup B) = P(A) + P(B)$$

for all sets $A, B \in \mathbf{F}$ such that

$$A \cap B \equiv \varnothing$$

EP 6 (*Annihilation*). $\{ v_i , w, - w ; v_i , w \in \Omega \ \&\ i \in I \} = \{ v_i; v_i \in \Omega \ \&\ i \in I \}$ for any element w from Ω.

Axiom EP6 shows that if w and $- w$ are taken (come) into one set, they annihilate one another. Having this in mind, we use two equality symbols = and ≡. The second symbol means equality of elements of sets. The second symbol also means equality of sets, when two sets are equal when they have exactly the same elements (Kuratowski and Mostowski, 1967). The equality symbol = is used to denote equality of two sets with annihilation, for example, $\{ w, - w \} = \varnothing$. Note that for sets, equality ≡ implies equality =.

For equality of numbers, we, as it is customary, use symbol =.

EP 7. (*Adequacy*) $A = B$ implies $P(A) = P(B)$ for all sets $A, B \in \mathbf{F}$.

For instance, $P(\{ w, - w \}) = P(\varnothing) = 0$.

EP 8. (*Non-negativity*) $P(A) \geq 0$, for all $A \in \mathbf{F}^+$.

It is known that for any set algebra **A**, the empty set Ø belongs to **A** and for any set field **B** in Ω, the set Ω belongs to **A** (Kolmogorov and Fomin, 1989).

Definition 2. The triad $(\Omega, \mathbf{F}, P)$ is called an *extended probability space*.

Definition 3. If $A \in \mathbf{F}$, then the number $P(A)$ is called the *extended probability* of the event A.

Let us obtain some properties of the introduced constructions.

Lemma 1. $\alpha(\Omega^+) \equiv -\Omega^+ \equiv \Omega^-$ and $\alpha(\Omega^-) \equiv -\Omega^- \equiv \Omega^+$.

Proof. By Axiom EP1, $\alpha(\Omega^+) \equiv -\Omega^+$ and $\alpha(\Omega^+) \supseteq \Omega^-$. As $\Omega \equiv \Omega^+ \cup \Omega^-$, Axiom EP1 also implies $\alpha(\Omega^+) \subseteq \Omega^-$. Thus, we have $\alpha(\Omega^+) \equiv \Omega^-$. The first part is proved.

The second part is proved in a similar way.

Thus, if $\Omega^+ = \{ w_i ; i \in I \}$, then $\Omega^- = \{ -w_i ; i \in I \}$.

As α is an involution of the whole space, we have the following result.

Proposition 1. α is a one-to-one mapping and $| \Omega^+ | = |\Omega^-|$.

Corollary 1. (*Domain symmetry*) $w \in \Omega^+$ if and only if $-w \in \Omega^-$.

Corollary 2. (*Element symmetry*) $- (- w) = w$ for any element w from Ω.

Corollary 3. (*Event symmetry*) $- (- X) \equiv X$ for any event X from Ω.

Lemma 2. $\alpha(w) \neq w$ for any element w from Ω.

Indeed, this is true because if $w \in \Omega^+$, then by Axiom EP1, $\alpha(w) \notin \Omega^+$ and thus, $\alpha(w) \neq w$. If $w \in \Omega^-$, then we may assume that $\alpha(w) = w$. However, in this case, $\alpha(v) = w$ for some element v from Ω^+ because by Axiom EP1, α is a projection of Ω^+ onto Ω^-. Consequently, we have

$$\alpha(\alpha(v)) = \alpha(w) = w$$

However, α is an involution, and we have $\alpha(\alpha(v)) = v$. This results in the equality

$$v = w$$

Consequently, we have $\alpha(v) = v$. This contradicts Axiom EP1 because $v \in \Omega^+$. Thus, lemma is proved by contradiction.

Proposition 2. $\Omega^+ \cap \Omega^- \equiv \varnothing$.

Proposition 3. $\mathbf{F}^+ \subseteq \mathbf{F}$, $\mathbf{F}^- \subseteq \mathbf{F}$ and $\mathbf{F} \subseteq \mathbf{F}^+ \cup \mathbf{F}^-$**.**

Corollary 1 implies the following result.

Proposition 4. $X \subseteq \Omega^+$ if and only if $- X \subseteq \Omega^-$.

Proposition 5. $\mathbf{F}^- \equiv \{X \in \mathbf{F};\ X \subseteq \Omega^- \} = \mathbf{F} \cap \Omega^-$.

Corollary 4. $\mathbf{F}^+ \cap \mathbf{F}^- \equiv \varnothing$.

Axioms EP6 implies the following result.

Lemma 3. $X \cup -X = \varnothing$ for any subset X of Ω.

Indeed, for any w from the set X, there is $-w$ in the set X, which annihilates w.

Let us define the union with annihilation of two subsets X and Y of Ω by the following formula:

$$X + Y \equiv (X \cup Y) \setminus [(X \cap -Y) \cup (-X \cap Y)]$$

Here the set-theoretical operation \ represents annihilation, while sets $X \cap -Y$ and $X \cap -Y$ depict annihilating entities.

Some properties of the new set operation + are the same as properties of the union $\cup$, while other properties are different. For instance, there is no distributivity between operations + and $\cap$.

Lemma 4. a) $X + X \equiv X$ for any subset X of Ω;

b) $X + Y \equiv X + Y$ for any subsets X and Y of Ω;

c) $X + \varnothing \equiv X$ for any subset X of Ω;

d) $X + (Y + Z) \equiv (X + Y) + Z$ for any subsets X , Y and Z of Ω;

e) $X + Y \equiv X \cup Y$ for any subsets X and Y of Ω^+ (of Ω^-);

Lemma 5. a) $Z \cap (X + Y) \neq Z \cap X + Z \cap Y$;

b) $X + (Y \cap Z) \neq (X \cap Y) + (X \cap Z)$.

Lemma 6. $A \cap B \equiv (A^+ \cap B^+) + (A^- \cap B^-)$ for any subsets A and B of Ω.

Indeed, as $A \equiv A^+ \cup A^-$ and $B \equiv B^+ \cup B^-$, we have

$$A \cap B \equiv (A^+ \cup A^-) \cap (B^+ \cap B^-) \equiv$$

$$(A^+ \cap B^+) \cup (A^+ \cap B^-) \cup (A^- \cap B^+) \cup (A^- \cap B^-) \equiv$$

$$(A^+ \cap B^+) + (A^- \cap B^-)$$

because $(A^+ \cap B^-) \equiv \varnothing$ and $(A^- \cap B^+) \equiv \varnothing$.

In a similar way, we prove the following results.

Lemma 7. $A \setminus B \equiv (A^+ \setminus B^+) + (A^- \setminus B^-)$ for any subsets A and B of Ω.

Lemma 8. $X \equiv X^+ + X^- = X^+ \cup X^-$ for any set X from **F**.

Lemma 9. $A + B \equiv (A^+ + B^+) + (A^- + B^-)$ for any sets X and Y from **F**.

Axioms EP6 and EP7 imply the following result.

Proposition 6. $\mathrm{P}(X + Y) = \mathrm{P}(X \cup Y)$ for any two events X and Y from Ω.

Lemma 10. $\mathrm{P}(\varnothing) = 0$.

Properties of the structure $\mathbf{F}^+$ are inherited by the structure **F** .

Theorem 1. (*Algebra symmetry*) If $\mathbf{F}^+$ is a set algebra (or set field), then **F** is a set field (or set algebra) with respect to operations + and ∩.

Proof. At first, we prove that $\mathbf{F}^-$ is a set algebra (or set field).

Let us assume that $\mathbf{F}^+$ is a set algebra and take two negative random events H and K from $\mathbf{F}^-$. By the definition of $\mathbf{F}^-$, $H = -A$ and $K = -B$ for some positive random events A and B from $\mathbf{F}^+$. Then we have

$$H \cap K = (- A) \cap (-B) = -(A \cap B)$$

As $\mathbf{F}^+$ is a set algebra, $A \cap B \in \mathbf{F}^+$. Thus, $H \cap K \in \mathbf{F}^-$.

In a similar way, we have

$$H \cup K = (- A) \cup (-B) = -(A \cup B)$$

As $\mathbf{F}^+$ is a set algebra, $A \cup B \in \mathbf{F}^+$. Thus, $H \cup K \in \mathbf{F}^-$.

By the same token, we have $H \setminus K \in \mathbf{F}^-$.

Besides, if $\mathbf{F}^+$ has a unit element E, then $-E$ is a unit element in $\mathbf{F}^-$.

Thus, $\mathbf{F}^-$ is a set algebra.

Now let us assume that $\mathbf{F}^+$ is a set field and $H \in \mathbf{F}^-$. Then by the definition of $\mathbf{F}^-$, $H = -A$ for a positive random event A from $\mathbf{F}^+$. It means that $C_{\Omega+}A = \Omega^+ \setminus A \in \mathbf{F}^+$. At the same time,

$C_{\Omega-}H = \Omega^- \setminus H = (-\Omega^+) \setminus (-A) = -(\Omega^+ \setminus A) = -C_{\Omega+}A$

As $C_{\Omega+}A$ belongs to $\mathbf{F}^+$, the complement $C_{\Omega-}H$ of H belongs to $\mathbf{F}^-$. Consequently, $\mathbf{F}^-$ is a set field.

Let us once more assume that $\mathbf{F}^+$ is a set algebra and take two random events A and B from $\mathbf{F}$. Then by Theorem 1, $\mathbf{F}^-$ is a set algebra. By Lemma 8, $A \equiv A^+ + A^-$ and $B \equiv B^+ + B^-$. By Axiom EP4, $A^+, B^+ \in \mathbf{F}^+$, $A^-, B^- \in \mathbf{F}^-$, while by Proposition 2, $A^+ \cap A^- \equiv \varnothing$, $B^+ \cap B^- \equiv \varnothing$, $A \equiv A^+ \cup A^-$, and $B \equiv B^+ \cup B^-$.

By Lemma 6, $A \cap B \equiv (A^+ \cap B^+) + (A^- \cap B^-)$. Thus, $(A \cap B)^+ \equiv A^+ \cap B^+$ and $(A \cap B)^- \equiv A^- \cap B^-$. As $\mathbf{F}^+$ is a set algebra, $(A \cap B)^+ \equiv A^+ \cap B^+ \in \mathbf{F}^+$. As it is proved that $\mathbf{F}^-$ is a set algebra, $(A \cap B)^- \equiv A^- \cap B^- \in \mathbf{F}^-$. Consequently, $A \cap B \in \mathbf{F}$.

By Lemma 7, $A \setminus B \equiv (A^+ \setminus B^+) + (A^- \setminus B^-)$. Thus, $(A \setminus B)^+ \equiv A^+ \setminus B^+$ and $(A \setminus B)^- \equiv A^- \setminus B^-$. As $\mathbf{F}^+$ is a set algebra, $(A \setminus B)^+ \equiv A^+ \setminus B^+ \in \mathbf{F}^+$. As it is proved that $\mathbf{F}^-$ is a set algebra, $(A \setminus B)^- \equiv A^- \setminus B^- \in \mathbf{F}^-$. Consequently, $A \setminus B \in \mathbf{F}$.

By Lemma 9, $A + B \equiv (A^+ + B^+) + (A^- + B^-)$. Thus, $(A + B)^+ \equiv A^+ + B^+$ and $(A + B)^- \equiv A^- + B^-$. As $\mathbf{F}^+$ is a set algebra, $(A + B)^+ \equiv A^+ + B^+ \equiv A^+ \cup B^+ \in \mathbf{F}^+$. As it is proved that $\mathbf{F}^-$ is a set algebra, $(A + B)^- \equiv A^- + B^- \equiv A^- \cup B^- \in \mathbf{F}^-$. Consequently, $A + B \in \mathbf{F}$.

Besides, if $\mathbf{F}^+$ has a unit element E, then $-E$ is a unit element in $\mathbf{F}^-$ and $E \cup -E$ is a unit element in $\mathbf{F}$.

Thus, $\mathbf{F}$ is a set algebra.

Now let us assume that $\mathbf{F}^+$ is a set field and $A \in \mathbf{F}$. Then as it is demonstrated above, $\mathbf{F}^-$ is a set field. By Lemma 8, $A \equiv A^+ + A^-$. By Proposition 2, $\Omega^+ \cap \Omega^- = \varnothing$, we have

$C_{\Omega}A = C_{\Omega+}A + C_{\Omega-}A$

Then $C_{\Omega+}A$ belongs to $\mathbf{F}^+$ as $\mathbf{F}^+$ is a set field and as it is proved in Theorem 1, $C_{\Omega-}A$ belongs to $\mathbf{F}^-$. Consequently, $C_{\Omega}A$ belongs to $\mathbf{F}^-$ and $\mathbf{F}^-$ is a set field.

Theorem is proved.

3. Negative interest rates and the problem of their modeling

Negative probabilities can help to model interest rates and interest rate derivatives. To show this, let us start with the equation

$$\text{Real interest rate} = \text{Nominal interest rate} - \text{Inflation rate} \qquad (1)$$

where

the nominal interest rate is the de facto rate, which is received by the lender and paid by the borrower in a financial contract. For example, the nominal interest rate is the rate, which the lender receives on a saving account or the coupon of a bond. From equation (1) we see that a real interest rate can easily be negative and in reality often is. For example, if the nominal interest rate on a savings account is 1% and the inflation rate is 3%, naturally, the real interest rate, i.e. the inflation adjusted rate of return for the lender is -2%.

3.1. Examples of negative nominal interest rates

However, in rare cases, also the nominal interest rate can be negative. An example of this would be that the lender gives money to a bank, and additionally gives pays the bank an interest rate. This happened in the 1970s in Switzerland. The lender had several motives

a) Switzerland is considered an extremely safe country to place capital
b) Investors were speculating on an increase of the Swiss franc
c) Some investors avoided paying taxes in their home country

Another example of negative nominal interest rates occurred in Japan in 2003. Banks lent Japanese Yen and were willing to receive a lower Yen amount back several days later. This means de facto a negative nominal interest rate. The reason for this unusual practice was that banks were eager to reduce their exposure to Japanese Yen, since confidence in the Japanese economy was low and the Yen was assumed to devalue.

Similarly, in the US, from August to November 2003, ‘repos’, i.e. repurchase agreements traded at negative interest rates. A repo is just a collateralized loan, i.e. the borrower of money gives collateral, for example a Treasury bond, to the lender for the time of the loan. When the loan is paid back, the lender returns the collateral. However, in 2003 in the US, settlement problems when returning the collateral occurred. Hence the borrower was only willing to take the risk of not having the collateral returned if he could pay back a lower amount than originally borrowed. This constituted a negative nominal interest rate.

A further example of the market expecting the possibility of negative nominal interest rates occurred in the worldwide 2008/2009 financial crisis, when strikes on options on Eurodollars Futures contracts were quoted above 100. A Eurodollar is a dollar invested at commercial banks outside the US. A Eurodollar futures price reflects the anticipated future interest rate. The rate is calculated by subtracting the Futures price from 100. For example, if the 3 month March Eurodollar future price is 98.5, the expected interest rate from March to June is 100 – 98.5 = 1.5, which is quoted in per cent, so 1.5%. In March 2009, option strikes on Eurodollar future contracts were quoted above 100 on the CME, Chicago Mercantile Exchange. This means that market participants could buy the right to pay a negative nominal interest on US dollars in the future if desired. The reason for this unusual behavior is that investors wanted to invest in the safe haven currency US dollar even if they had to pay for it.

3.2. Modeling interest rates

In finance, interest rates are typically modeled with a geometric Brownian motion,

$$\frac{dr}{r} = \mu_r \, dt + \sigma_r \, \varepsilon\sqrt{dt} \qquad (2)$$

dr : change in the interest rate r;

μ_r : drift rate, which is the expected growth rate of r, assumed non-stochastic and constant

dt : infinitely short time period

σ_r : expected volatility of rate r, assumed non-stochastic and constant

ε : random drawing from a standardized normal distribution. All drawings at times t are independent from each other.

In equation (2), the first term on the right hand side gives the average growth rate of r. The second term on the ride side adds stochasticity to the process via ε, i.e. provides the distribution around the average growth rate. Importantly, from equation (1) we can observe that the relative change dr/r is normally distributed, since ε is normally distributed. If the relative change of a variable is normally distributed, it follows that the variable itself is log-normally distributed with a pdf

$$\frac{1}{x\,\sigma\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} \qquad (3)$$

In the equation (3), μ and σ are the mean and standard deviation of ln(x) respectively. Figure 1 shows a log-normal distribution.

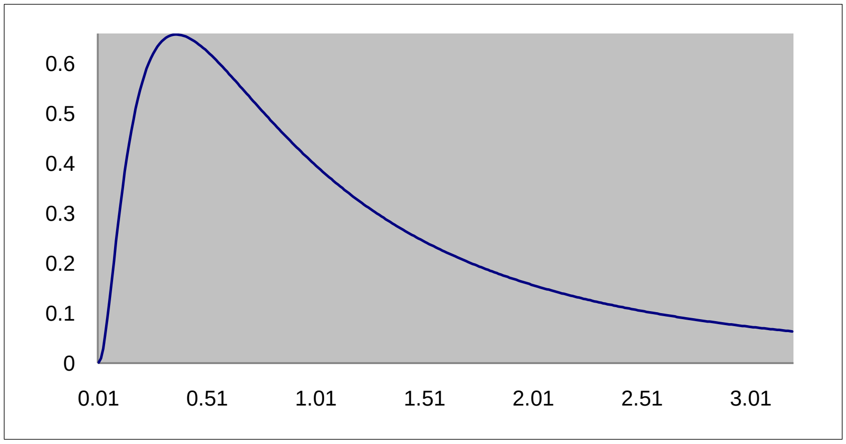

Figure 1 Log-normal distribution with $\mu = 0$ and $\sigma = 1$.

The logarithm of a negative number is not defined, hence with the pdf equation (3), negative values of interest rates cannot be modeled. However, as discussed above, negative interest rates do exist in the real financial world. Here negative probabilities come into play. We will explain this with options on interest rates.

4. How negative probabilities allow more adequate interest rate modeling

4.1. Modeling interest rate options

Two main types of options are call options and put options.

A *call option* reflects the right but not the obligation to pay a strike price and receive an underlying asset.

A *put option* reflects the right but not the obligation to receive a strike price and deliver an underlying asset.

Let's derive the equations used in financial practice to value calls and puts. From equation (2), applying Ito's lemma, we derive the famous 1997 Nobel Prize rewarded work of Black, Scholes and Merton, which resulted in the PDE[1]

$$D = \frac{\partial D}{\partial t}\frac{1}{i} + \frac{\partial D}{\partial S}S + \frac{1}{2}\frac{\partial^2 D}{\partial S^2}\frac{1}{i}\sigma^2 S^2 \qquad (4)$$

where

D : financial derivatives as for example a call option or a put option

i: discount rate

S : modeled variable

σ : volatility of S

One equation that satisfies the PDE (4) is the Black-Scholes-Merton equation for a call and a put[2]. For a call option, we have

[1] For a proof see www.dersoft.com/BSMPDEgeneration.pptx
[2] For a proof, see www.dersoft.com/bspdeproof.doc

$$C = S_0\, N(d_1) - Ke^{-iT}\, N(d_2) \;\text{ with }\; d_1 = \frac{\ln(\frac{S_0}{Ke^{-iT}}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \;\text{ and }\; d_2 = d_1 - \sigma\sqrt{T} \qquad (5)$$

where

N(x) : cumulative probability of a standard normal distribution at x

K : Strike, i.e. the price that the buyer may pay at option maturity T to receive the underlying asset

4.2. Caps and Floors

For interest rate options, a term structure of interest rates exists (i.e. market given interest rates for different maturities exist). This property is utilized when valuing interest rate options in the model Black 1976 model. We will first discuss an interest rate option contract on short term interest rates, i.e. Caps and Floors. A Cap consists of several Caplets. A Caplet is the option but not the obligation to pay an interest rate r_K at option maturity t_x. A Floor consists of several Floorlets. A Floorlet is the option but not the obligation to receive an interest rate r_K at option maturity t_x.

A Cap is typically used as an insurance against rising interest rates: If a borrower pays a floating interest rate on his loan, he can protect himself against rising interest rates by buying a Cap. Conversely, a Floor is typically used as an insurance for decreasing interest rates. If an investor receives a floating interest rate on a bond, the investor can protect himself against decreasing interest rate by buying a Floor.

Using forward interest rates r_f, equation (5) becomes

$$\text{Caplet}_{\,t_s,t_l} = m\, e^{-r_l\, t_l}\; \{r_f\, N(d_1) - r_k\, N(d_2)\} \;\text{ with }\; d_1 = \frac{\ln(\frac{r_f}{r_k}) + \frac{1}{2}\sigma^2 t_x}{\sigma\sqrt{t_x}} \;\text{ and }\; d_2 = d_1 - \sigma\sqrt{t_l}$$

(6)

$$\text{Floorlet}_{t_s,t_l} = m\ \text{PA}\ e^{-r_l t_l}\ \{r_k\, N(-d_2) - r_f\, N(-d_1)\} \qquad (7)$$

where d_1 and d_2 are defined as in (6)

Caplet : option on an interest rate from time t_s to time t_l, $t_l > t_s$, i.e. the right but not the obligation to pay the rate r_K at t_l.

Floorlet : option on an interest rate from time t_s to time t_l, $t_l > t_s$, i.e. the right but not the obligation to receive the rate r_K at option maturity t_l.

m : time between t_l and t_s, expressed in years

t_x: option maturity, $t_x \leq t_s < t_l$

r_f : forward interest rate, derived as $r_{f_{t_s,t_l}} = \left(\frac{df_{ts}}{df_{tl}} - 1\right)\left(\frac{1}{t_l - t_s}\right)$ where df is a discount factor, i.e.

$df_{ty} = 1/(1+r_y)$.

r_K : strike rate i.e. the interest rate that the Caplet buyer may pay and the Floorlet buyer may receive at option maturity t_x.

For more details, see Meissner 1998.

4.3. Applying negative probabilities to Caplets and Floorlets

Our original problem is that the market applied log-normal distribution, which is underlying the valuation of interest rate derivatives, cannot model negative interest rates. Several solutions to this problem are possible.

a) We can model interest rates with an entirely different distribution as for example the normal distribution, which allows negative interest rates. This is done by Vasicek (1977), Ho and Lee (1986), and Hull and White (1990). However, empirical data shows that interest rate distribution behaves far more log-normal than normal. Thus, the suggested solutions do not correctly reflect the reality.

b) We can add a location parameter to the log-normal distribution. Hence equation (3)

$$\frac{1}{x\,\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} \text{ becomes } \frac{1}{(x-\alpha)\,\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{\ln(x-\alpha)-\mu}{\sigma}\right)^2}$$

, where α is the location parameter. For $\alpha > 0$, the log-normal distribution is shifted to the left. As a result, the probability distribution acquires negative values. This is impossible in the conventional probability theory but fits well into extended probability theory. At the same time, probability distributions that take negative values are important for practice, allowing to model negative interest rates.

c) A further way to model options on negative interest rates is to apply negative probabilities to equations (6) and (7). We add a parameter β to equations (6) and (7)

$$\text{Caplet}_{t_s,t_l} = m\ e^{-r_l t_l}\ \{r_f\,[N(d_1)-\beta] - r_k\,[N(d_2)-\beta]\} \qquad \beta \in \Re \quad (8)$$

$$\text{Floorlet}_{t_s,t_l} = m\ e^{-r_l t_l}\ \{r_k\,[N(-d_2)-\beta] - r_f\,[N(-d_1)-\beta]\} \qquad (9)$$

This also brings us to negative probabilities. If the strike r_K is smaller than the forward rate s_f, for a positive β, negative probabilities may emerge, i.e. $N(d_1)-\beta$, $N(d_2)-\beta$, $N(-d_2)-\beta$ and $N(-d_1)-\beta$ may become negative. Thus, appearance of negative probabilities depends on the value of β and the option input parameters r_K, r_F, r_l, and σ. The higher the value of β, the more likely it is that negative probabilities will emerge.

Importantly, applying negative probabilities in equations (8) and (9) decreases the value of the Caplet and increase the value of the Floorlet. This is an adequate result since it adjusts the option prices for the possibility of negative interest rates. The magnitude of the parameter β, that a trader applies, reflects a trader's opinion on the probability of negative rates. A trader will use more extreme β-values if he/she believes strongly in the possibility of negative interest rates, vice versa.

4.4. Swapoptions

Another popular type of interest rate option is a Swapoption. Two types exist, Payers and Receivers. A Payers Swapoption is the right but not the obligation to pay a fixed interest rate, i.e. 5%, and receive a floating interest rates, e.g. 3-month Libor[3]. A Receivers Swapoption is the right but not the obligation to receive a fixed interest rate, i.e. 5%, and pay a floating interest rates, e.g. 3-month Libor. Similar to Caps and Floors, Swapoptions are typically used to protect against interest rate volatility. For example, a company is bidding on an investment project and is concerned about rising interest rates until the bidding is decided. The company can enter into a Payers Swapoption. If the company wins the bid and interest rates have risen, the Payers will allow the company to pay the lower strike rate, which was agreed in the Swapoption.

A Payers Swapoption can be valued by modifying equation (5). This gives us an equation for Payers Swapoption (PSWO):

$$\text{PSWO} = \sum_{i=1}^{n} \frac{PA_i (t_i - t_{i-1})}{(1+sr_f)^{(p_{t_i} - T)}} e^{-rT} \left(sr_f \; N(d_1) - r_k \; N(d_2) \right) \tag{10}$$

where

PA_i is the principal amount for the period from t_{i-1} to t_i

sr_f is the forward swap rate, for the period from t_{i-1} to t_n.

$sr_f = (df_{t_{i-1}} - df_{t_n}) \,/\, \sum_{i=1}^{n} df_{ti}(t_i - t_{i-1})$ and df is a standard discount factor, i.e. $df_l = 1/(1+r_l)$.

$$d_1 = \frac{\ln(\frac{sr_f}{r_k}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

p_{t_i} is the payment date of the fixed cash flows

Similarly, the Receiver Swapoption (RSWO) can be valued as

[3] Libor stands for London Interbank offered rate. It is determined every business day at 11 AM London time.

$$\text{RSWO} = \sum_{i=1}^{n} \frac{PA_i (t_i - t_{i-1})}{(1+sr_f)^{(p_{t_i} - T)}} \, e^{-rT} \left(r_k \; N(-d_2) - sr_f \; N(-d_1) \right) \qquad (11)$$

For more details see Meissner, 1998.

We once more encounter a problem because the market applied log-normal distribution, which is underlying the valuation of interest rate derivatives, cannot model negative interest rates. To solve this problem, we apply the approach used for Caps and Floors to Payers and Receivers Options.

In this case, it is again possible as in (a), to use an entirely different distribution as the normal distribution or as in (b), to add a location parameter to the log-normal distribution. A third possibility is to apply negative probabilities once more. We can add a parameter γ to equations (10) and (11):

$$\text{PSWO} = \sum_{i=1}^{n} \frac{PA_i (t_i - t_{i-1})}{(1+sr_f)^{(p_{t_i} - T)}} \, e^{-rT} \left(sr_f \; [N(d_1)-\gamma] - r_k \; [N(d_2)-\gamma] \right) \qquad (12)$$

and

$$\text{RSWO} = \sum_{i=1}^{n} \frac{PA_i (t_i - t_{i-1})}{(1+sr_f)^{(p_{t_i} - T)}} \, e^{-rT} \left(r_k \; [N(-d_2)-\gamma] - sr_f \; [N(-d_1)-\gamma] \right) \qquad (13)$$

We derive equivalent results as we did for Caps and Floors. If the strike r_K is smaller than the forward rate sr_f , for a positive γ, negative probabilities may emerge, i.e. $N(d_1)-\gamma$, $N(d_2)-\gamma$, $N(-d_2)-\gamma$ and $N(-d_1)-\gamma$ may become negative. Thus, appearance of negative probabilities depends on the value of γ and the option input parameters r_K, sr_f, r, T and σ. The higher the value of γ, the more likely it is that negative probabilities will emerge.

Importantly, applying negative probabilities decreases the value of the Payers Swapoption and increases the value of the Receivers Swapoption. This is the desired result since it adjusts the option prices for the possibility of negative interest rates. The magnitude of parameter γ that a trader applies, reflects a trader's opinion on the probability of negative rates. A trader will use more extreme γ-values if he/she believes strongly in the possibility of negative interest rates, vice versa.

5. Concluding Summary

We have defined extended probabilities, which include negative probabilities, and derived their general properties. Then we have applied extended probabilities to financial modeling. We have shown that negative nominal interest rates have occurred several times in the past in financial practice, as in the 2008/2009 global financial crisis. This is inconsistent with the conventional theoretical models of interest rates, which typically apply a log-normal distribution. In particular, when Caps, Floors and Swapoption are valued in a log-normal Black-Scholes-Merton framework, then the probability of negative interest rates is zero. Here negative probabilities come into play. We have shown that integrating negative probabilities in the Black-Scholes-Merton framework allows to consistently model negative nominal interest rates, which exist in financial practice.

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